Systems of differential equations Handout

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This handout is meant to give you a couple more example of all the techniques discussed in chapter 9, to counterbalance all the dry theory and complicated applications in the differential equations book! Enjoy! :)

Note: Make sure to read this carefully! The methods presented in the book are a bit strange and convoluted, hopefully the ones presented here should be easier to understand!

1 Systems of differential equations

Find the general solution to the following system:

$$\begin{cases} x_1'(t) = & -x_1(t) - x_2(t) + 3x_3(t) \\ x_2'(t) = & x_1(t) + x_2(t) - x_3(t) \\ x_3'(t) = & -x_1(t) - x_2(t) + 3x_3(t) \end{cases}$$

First re-write the system in matrix form:

$$\mathbf{x}' = A\mathbf{x}$$

Where:

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \qquad A = \begin{bmatrix} -1 & -1 & 3 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

Now diagonalize A: $A = PDP^{-1}$, where:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

Note: To find the eigenvalues, solve $det(A - \lambda I) = 0$. You should get $\lambda = 1, 2, 0$. The diagonal entries of D are $\lambda = 1, 2, 0$. Then, for each eigenvalue, find a basis for $Nul(A - \lambda I)$. The the columns of P are the eigenvectors you found.

Then use the following fact:

Fact: For each eigenvalue λ and eigenvector $\mathbf{v}$ you found, the corresponding solution is $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$

Hence here, the solution is:

$$\mathbf{x}(t) = Ae^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + Be^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

(**Note:** Here $e^{0t} = 1$)

1.1 Aside: Why does this work?

Suppose you want to solve $\mathbf{x}' = A\mathbf{x}$, since $A = PDP^{-1}$, this becomes:

$$\mathbf{x}' = PDP^{-1}\mathbf{x}$$

So:

$$\mathbf{x}' = PD\left(P^{-1}\mathbf{x}\right)$$

Now let $\mathbf{y} = P^{-1}\mathbf{x}$, so $\boxed{\mathbf{x} = P\mathbf{y}}$ (remember Peyam, not Pexam). Then the above becomes:

$$\mathbf{x}' = PD\mathbf{y}$$

$$P^{-1}\mathbf{x}' = D\mathbf{y}$$

But P^{-1} is like a constant, so it gets inside the derivative!

$$\left(P^{-1}\mathbf{x}\right)' = D\mathbf{y}$$

Finally, use $\mathbf{y} = P^{-1}\mathbf{x}$, and you get:

$$\mathbf{y}' = D\mathbf{y}$$

Now solve the system: $\mathbf{y}' = D\mathbf{y}$, **which is easier to solve:**

$$\begin{cases} y_1'(t) = & y_1(t) \\ y_2'(t) = & 2y_1(t) \\ y_3'(t) = & 0 \end{cases}$$

Which gives you:

$$\begin{cases} y_1(t) = & Ae^t \\ y_2(t) = & Be^{2t} \\ y_3(t) = & Ce^{0t} = C \end{cases}$$

Finally, use $\mathbf{x} = P\mathbf{y}$ to get:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} Ae^t \\ Be^{2t} \\ C \end{bmatrix} = Ae^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + Be^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Note: The matrix:

$$X(t) = \begin{bmatrix} e^t & e^{2t} & 1 \\ e^t & 0 & -1 \\ e^t & e^{2t} & 0 \end{bmatrix}$$

(where you essentially ignore the constants A, B, C) is called a **fundamental matrix** for the system.

2 Complex eigenvalues

2.1 Solve the system $\mathrm{x}' = A\mathrm{x}$, where:

$$A = \begin{bmatrix} -1 & -2 \\ 8 & -1 \end{bmatrix}$$

Eigenvalues of A: $\lambda = -1 \pm 4i$.

From now on, only consider one eigenvalue, say $\lambda = -1 + 4i$.

A corresponding eigenvector is $\begin{bmatrix} i \\ 2 \end{bmatrix}$

Now use the following fact:

Fact: For each eigenvalue λ and eigenvector $\mathbf{v}$ you found, the corresponding solution is $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$

Hence, one solution is:

$$\mathbf{x}(t) = e^{(-1+4i)t} \begin{bmatrix} i \\ 2 \end{bmatrix}$$

Now split into real and imaginary parts and multiply everything out and group everything back into real and imaginary parts to get:

$$\begin{aligned}
\mathbf{x}(t) &= \left(e^{-t}\cos(4t) + ie^{-t}\sin(4t)\right)\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\
&= \left(e^{-t}\cos(4t)\begin{bmatrix} 0 \\ 2 \end{bmatrix} - e^{-t}\sin(4t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + i\left(e^{-t}\sin(4t)\begin{bmatrix} 0 \\ 2 \end{bmatrix} + e^{-t}\cos(4t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)
\end{aligned}$$

Hence the general solution is:

$$\mathbf{x}(t) = A\left(e^{-t}\cos(4t)\begin{bmatrix} 0 \\ 2 \end{bmatrix} - e^{-t}\sin(4t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + B\left(e^{-t}\sin(4t)\begin{bmatrix} 0 \\ 2 \end{bmatrix} + e^{-t}\cos(4t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

3 Undetermined coefficients

3.1 Solve $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{f}(t) = \begin{bmatrix} e^t + \cos(t) \\ 4e^t \end{bmatrix}$$

As usual, $\mathbf{x}(t) = \mathbf{x_0}(t) + \mathbf{x_p}(t)$, where:

- $\mathbf{x_0}(t)$ is the general solution to $\mathbf{x}' = A\mathbf{x}$
- $\mathbf{x_p}(t)$ is *one particular* solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$

To find $\mathbf{x_0}$, use the techniques discussed in 1.

To find $\mathbf{x_p}$, use:

Undetermined coefficients:

First group the terms in $\mathbf{f}$ which 'look alike':

$$\mathbf{f}(t) = \begin{bmatrix} e^t \\ 4e^t \end{bmatrix} + \begin{bmatrix} \cos(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(t)$$

Now guess:

$$\mathbf{x_p} = \mathbf{a}e^t + \mathbf{b}\cos(t) + \mathbf{c}\sin(t)$$

where $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Now plug in $\mathbf{x_p}$ into the equation $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, and solve for $\mathbf{a}, \mathbf{b}, \mathbf{c}$

Remarks:

1) Notice how similar this is to our usual way of doing undetermined coefficients! The only difference here is that $\mathbf{a}$ is a vector instead of a number!

2) Remember to always put a $\sin(t)$ term whenever you see a $\cos(t)$ term and vice-versa!

3) The same rule about adding a t or not holds in this case too (but it's very rare).

4 Variation of parameters

4.1 Find the general solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, where:

$$A = \begin{bmatrix} -1 & -1 & 3 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix} \qquad \mathbf{f}(t) = \begin{bmatrix} e^t \\ ln(t) \\ \tan(t) \end{bmatrix}$$

As usual, $\mathbf{x}(t) = \mathbf{x_0}(t) + \mathbf{x_p}(t)$.

We already found $\mathbf{x_0}(t)$ in the first example:

$$\mathbf{x_0}(t) = A \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix} + B \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix} + C \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

To find $\mathbf{x_p}(t)$, use:

Variation of Parameters: Suppose

$$\mathbf{x_p}(t) = v_1(t) \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix} + v_2(t) \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix} + v_3(t) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Consider the (pre)-Wronskian (or fundamental matrix):

$$\widetilde{W}(t) = X(t) = \begin{bmatrix} e^t & e^{2t} & 1 \\ e^t & 0 & -1 \\ e^t & e^{2t} & 0 \end{bmatrix}$$

(essentially put all the vectors you found in one matrix)
And solve:

$$\widetilde{W}(t) \begin{bmatrix} v_1'(t) \\ v_2'(t) \\ v_3'(t) \end{bmatrix} = \begin{bmatrix} e^t \\ ln(t) \\ \tan(t) \end{bmatrix}$$

That is:

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \\ v_3'(t) \end{bmatrix} = \left(\widetilde{W}(t)\right)^{-1} \begin{bmatrix} e^t \\ ln(t) \\ \tan(t) \end{bmatrix}$$

This gives you v_1', v_2', v_3'.

To get v_1, v_2, v_3, integrate the equations you found.

And finally, to get $\mathbf{x_p}$, use:

$$\mathbf{x_p}(t) = v_1(t) \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix} + v_2(t) \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix} + v_3(t) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

And hence $\mathbf{x}(t) = \mathbf{x_0}(t) + \mathbf{x_p}(t)$.

Note: The following formula might come in handy:

$$\text{If} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

5 Matrix exponential

5.1 Find e^{At}, where:

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

Eigenvalues of A: $\lambda = -2$, with multiplicity 3.

IMPORTANT: The following technique works only in this case (where we have one eigenvalue with full multiplicity). For all the other cases, use the next example.

Then:

$$e^{At} = e^{-2t}\left(I + (A+2I)t + (A+2I)^2\frac{t^2}{2!}\right) = \begin{bmatrix} e^{-2t} & 0 & 0 \\ 4te^{-2t} & e^{-2t} & 0 \\ te^{-2t} & 0 & e^{-2t} \end{bmatrix}$$

Note: If λ had multiplicity 2, we would stop at $(A+2I)t$. But if it had multiplicity 4, we would add a $(A+2I)^3\frac{t^3}{3!}$ term.

General method:

5.2 Find e^{At}, where:

$$A = \begin{bmatrix} 16 & -35 \\ 6 & -13 \end{bmatrix}$$

Diagonalize A: $A = PDP^{-1}$, where:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$$

Then:

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{t} \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 15e^{2t} - 14e^{t} & 35e^{t} - 35e^{2t} \\ 6e^{2t} - 6e^{t} & 15e^{t} - 14e^{2t} \end{bmatrix}$$

Point: The general solution to $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x}(t) = e^{At}\mathbf{c}$, where $\mathbf{c}$ is a constant vector!

Here, we get:

$$\mathbf{x}(t) = A \begin{bmatrix} 15e^{2t} - 14e^{t} \\ 6e^{2t} - 6e^{t} \end{bmatrix} + B \begin{bmatrix} 35e^{t} - 35e^{2t} \\ 15e^{t} - 14e^{2t} \end{bmatrix}$$